Quantum Mechanics and Human Decision Making

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Abstract

In physics, at the beginning of the twentieth century it was recognized that some experiments could not be explained by the conventional classical mechanics but the same could be explained by the newly discovered quantum theory. It resulted into a new mechanics called quantum mechanics that revolutionized the scientific and technological developments. Again at the beginning of the twenty-first century, it is being recognized that some experiments related with the human decision making processes could not be explained by the conventional classical decision theory but the same could be explained by the models based on quantum mechanics. It is now recognized that we need quantum mechanics in psychology as well as in economics and finance. In this paper we attempt to advance and explain the present understanding of applicability of quantum mechanics to the human decision making processes. Using the postulates analogous to the postulates of quantum mechanics, we show the derivation of the quantum interference equation to illustrate the quantum approach. The explanation of disjunction effect experiments of Tversky and Shafir(1992) has been chosen to demonstrate the necessity of a quantum model. Further to suggest the possibility of application of the quantum theory to the business related decisions, some terms such as price operator, state of mind of the acquiring firm, etc. are introduced and discussed in context of the merger/acquisition of business firms. The possibility of the development in the areas such as quantum finance, quantum management, application of quantum mechanics to the human dynamics related with health care management, etc. is also indicated.

Key words: Quantum decision model, quantum interference, disjunction effect, human decision making, quantum information processing, merger/acquisition of business firms, cognitive science, two-stage gambling experiment, the sure-thing principle, Tversky-Shafir experiments.

1. Introduction

Under some conditions, light waves reaching from two sources at a point can interfere to cancel each other resulting into the darkness at that point. Such strange behavior of light waves was well known even before the discovery of quantum mechanics. The development of quantum mechanics in early 20th century generalized such wave nature and interference phenomenon to electrons and other particles. In addition to quantum interference, quantum mechanics also yields many other peculiar results such as uncertainty principle, quantum nature of light, quantum theory of measurement, tunneling, etc. which are in contradiction with our common sense. A famous physicist, Bohr who won the 1922 Noble prize of physics chiefly for his work on atomic structure, once remarked, "*If quantum mechanics hasn't profoundly shocked you, you haven't understood it yet*."

Despite its strange behavior, the quantum mechanics is considered as the most successful theory of physics. Stenholm and Suominen (2005) in their book on quantum approach to informatics write: "*Quantum theory has turned out to be the most successful theory of physics. ... Without the understanding offered by quantum theory, our ability to build integrated circuits and communication devices would not have emerged.*" A large number of scientists (e.g., Planck, Einstein, Bohr, de Broglie, Heisenberg, Schrodinger, Born, Dirac, Pauli, Pauling) have been awarded Noble prizes for their contributions related with quantum mechanics. In various walks of modern science and technology including electronics, nuclear technology, nano-technology, femto-chemistry, molecular biology, cosmology, high energy physics, quantum mechanics is valuable and indispensible.

In recent years, one notes a growing interest in the application of quantum mechanics to areas such as quantum cryptography (e.g., Bennett and Brassard 1984; Bennett et. al. 1992; Chung et. al. 2008) as well as quantum computation (e.g., Shor 1997; Lo et. al. 2000; Hand 2009). As regards the application of quantum mechanics beyond physical sciences, Bohr (1929) attempted to show the similarity between the mental processes and the quantum mechanical phenomena. In his writings, he also discussed the similarities between quantum mechanics and the functions of the brain (e.g., Bohr 1933). In recent decades, there have been various notable attempts to ascribe the quantum mechanical properties to brain, mind, and consciousness (e.g., Chalmers 1996: Lockwood1989; Penrose 1989; Penrose et. al. 2000; Pessa and Vitiello 2003; Satinover 2001).

Recently there has been some new work to explore applicability of quantum models in better understanding nuances of human decision making. The purpose of this paper is to introduce and explain the quantum concepts through simple terms and notations, to apply these concepts in better understanding the recent applications of quantum mechanics to human decision making, and suggest applicability of models based on quantum mechanics to some new areas of research.

1.1 Quantum mechanics and human decision making

Regarding human mind, economics Nobel Laureate, Herbert Simon wrote in collaboration with Newell (Simon and Newell, 1958): *"The revolution in heuristic problem solving will force man to consider his role in a world in which his intellectual power and speed are outstripped by the intelligence of machines. Fortunately, the new revolution will at the same time give him a deeper understanding of the structure and working of his own mind.*" It is interesting to note the significance attached by these authors to quantum mechanics in the same paper in these words: "*In dealing with the ill-structured problems of management we have not had the mathematical tools we have needed – we have not had 'judgment mechanics' to match quantum mechanics*." The expectations of Simon and Newell expressed half a century back regarding the necessity of understanding of our own mind and a mechanics of the decision making process are not yet fulfilled. But we are now gaining momentum in the direction of understanding the human decision processes even through quantum mechanics. Thus the objective of this paper is to introduce these concepts and review the recent progress to stimulate more exploratory research on applications of quantum mechanics concepts in decision making.

Kahenman, Tversky, and Shafir have made notable contributions in the area of judgment under uncertainty and the influence of heuristics and biases on the cognitive system (Tversky and Kahenman 1974; Tversky and Shafir 1992; Shafir and Tversky 1992). The significance of the work is attested by the fact that Kahenman was awarded the Nobel prize in 2002. Results of several experiments related to the judgment under uncertainty as noted by Tversky and Shafir (Tversky and Shafir 1992; Shafir and Tversky 1992) in the area of human psychology cannot be explained by the classical statistics. The disjunction effect experimentally observed by Tversky and Shafir (1992) is a typical example of the intricacies of human mind which cannot be understood by the classical decision theory. For example, in an experiment of Tversky and Shafir (1992), a participant is offered to play a gamble (by tossing a coin) with a 50% chance of winning $200 and a 50% chance of losing $100. After the first play, the participant is offered to play the second identical game with or without the knowledge of the outcome of the first gamble. It has been observed that a majority of participants are ready to accept the second gamble after knowing that they have won the first one, and a majority of participants are also ready to accept the second gamble after knowing that they have lost the first one, but only a small fraction of participants are ready to accept the second gamble if they do not know the outcome of the first gamble. The question arises: if they prefer to accept the second gamble in case they win or lose the first gamble, then according to the sure-thing principle of Savage (1954), they should prefer to accept the second gamble even when they do not know the outcome of the first gamble. But the experiment contradicts such logical expectations. Why? We cannot get the answer of this 'why' from the conventional (classical) theories. Such a violation of the sure-thing principle of Savage (1954) has also been observed by Tversky and Shafir (1992) in another experiment related with the buying of an attractive vacation package.

The successful studies to explain some of these paradoxes by incorporating mathematical equations related with quantum mechanics into the psychology (Busemeyer et. al. 2006; Pothos and Busemeyer 2010; Khrennikov 2009; Yukalov and Sornette 2009a) clearly reveal that some aspects of the human behavior can be explained by quantum mechanical equations, but not by the classical mechanics. It may be noted that classical mechanics and quantum mechanics differ ideologically as well as mathematically; and for macro-systems the approximate form of mathematical equations of quantum mechanics agrees with the equations of classical mechanics.

If the decision making process of human mind may follow the probabilistic behavior of quantum mechanics, then one can expect the applicability of the same in other areas, which are directly affected by human decision making. Thus it is not surprising that researchers in economics and finance have explored application of quantum mechanics (Kondratenko 2005; Baaquie 2004; Baaquie 2009a). The application of quantum mechanics to economics and finance can be seen in various areas, such as price dynamics model (Choustova 2007), stock price (Schaden 2003; Bagarello 2009), interest rate (Baaquie 2009b), incorporation of private information (Ishio and Haven 2009; Haven 2008), etc. As an example of the value of quantum mechanics in the field of economics, one can refer to the studies conducted by Segal and Segal (1998). In this study they consider quantum effects to explain extreme irregularities in the evolution of prices in financial markets. In the concluding paragraph of this study Segal and Segal (1998) write: " *The quantum extension of Black-Scholes-Merton theory provides a rational, scientifically economical, and testable model toward the explanation of market phenomena that show greater extreme deviations than would be expected in classical theory*…"

By the phrase 'understanding human decision process through quantum mechanics' we mean the application of some aspects, such as mathematical framework, of quantum mechanics. For example, we may consider some states of mind in an abstract space which mathematically behave as the quantum states in the Hilbert space (Von Neumann 1983; Messiah 1961), and the decision making process as a process statistically governed by the formulation based on the postulates of quantum mechanics. This, however, does not mean that human mind becomes a quantum mechanical object. Just as a quantum description of electrons, light quanta, etc. require the necessity of a constant known as Planck's constant ($h = 6.626 \times 10^{-34}$ Joule-Second), we do not need Planck's constant for explaining the above mentioned disjunction effect or other paradoxes of psychology. Likewise, in quantum mechanics, Schrodinger's time independent and time dependent wave equations contain Planck's constant , but in the corresponding equations of quantum dynamics of human decision making (Busemeyer et al.2006; Pothos and Busemeyer 2010), this constant occurring in the equations of quantum mechanics is replaced by another parameter.

1.1.1 Application of quantum models to disjunction effect and other decisions

While explaining the disjunction effect and other paradoxes of psychology with the help of quantum models, Khrennikov (2009) assumes the effect of quantum interference in the form of an equation that has an adjustable parameter named as the coefficient of interference. Yukalov and Sornette (2008, 2009a, 2009b, 2009c) provide a detailed theory named as quantum decision theory (QDT). Using the postulates analogous to the postulates of quantum mechanics they derive the quantum interference equations that relate various experimental probabilities. Thus the equation related with quantum interference assumed by Khrennikov (2009) has been derived by Yukalov and Sornette (2009a).

To explain the same disjunction effect, Pothos and Busemeyer (2010) consider the evolution of the state of mind using an equation analogous to Schrodinger's time dependent wave equation of quantum mechanics. The duration of time and the interaction parameters have been considered as adjustable parameters. For comparison, they also study the evolution of the state of mind using the equivalent Markov (classical) model, and conclude that the classical model is unable to explain the experimentally observed violations of the sure-thing principle of Savage (1954) whereas the quantum model can explain.

In a quantum decision model being discussed in detail in section 2, we employ various aspects of quantum decision theory of Yukalov and Sornette (2009a) but consider more general and simpler kind of operators to derive the same quantum interference equation as derived by Yukalov and Sornette (2009a) so that the range of applicability may widen and it becomes easier to apply to other related problems. Further, to demonstrate the possibility of application of the quantum approach to other decision problems, we consider an example of problem of merger of two business firms. It may be a long way to arrive at a successful and valuable outcome of the application of a quantum model to the problem of merger of two business firms. However, here we shall simply introduce the problem just to familiarize with the notations and application of the quantum models in this area.

Thus the purpose of this paper is to serve as a tutorial paper by illustrating current and future potential applications of quantum concepts to human decision making. In section 2, the mathematical details of a sample quantum model are described. Application of the quantum approach in explaining the disjunction experiments, and the possibility of its application to the business related problems together with the discussion of the related studies in this area are presented in section 3. The final section provides the summary and concluding remarks.

2. Mathematical details of a sample decision model

To simplify, we describe the model with reference to a practical example. We consider the two-stage gambling experiment of Tversky and Shafir (1992). In this experiment, subjects are first offered to play a gamble (by tossing a coin) with a 50% chance of winning \$200 and a 50% chance of losing \$100. After the first play, the subjects are offered to play the identical game with or without the knowledge of winning or losing the first gamble. We introduce several probabilities and the related notations associated with this experiment in Table 1. In this section, we shall describe the quantum decision model to explain the results of this experiment which cannot be explained by the conventional (classical) methods.

For an understanding of the basic postulates of quantum mechanics, we consider an analogy: Suppose we ask our banker to provide us an information regarding the amount of interest earned by us. To answer this question, the bank teller would first open our account that would contain all information regarding our deposits, check withdrawals, etc. The details regarding our account in the register or the computer screen would give the present status of our account. To answer our query regarding the amount of interest, the teller would perform some operations/calculations related to the amount of interest. Similarly, in quantum mechanics, analogous to the details of our account, there exists a state function or a state ket or a state vector or simply a state of the system for the system under consideration; and corresponding to 'the amount of interest' in our analogy, in quantum mechanics we have an operator. For example, in physics there are operators corresponding to energy, momentum, position, etc., and in the quantum decision model we will see that there can be operators like price operator, buy operator, pass operator, win operator, accept operator, etc. Finally, from some operations of the operators on the state of the system we would get the desired result just as a bank teller arrives at the amount of interest earned.

The example above describes the method of getting the desired information in the framework of quantum mechanics. We present such a framework in Table 2. We describe the postulates of quantum mechanics (Messiah 1961; White 1966; Agrawal 1989) and also introduce those for the quantum decision model. One would note that there is one-to-one correspondence between these two sets of postulates. Just like the existence of 'our account' in the above mentioned example, postulate #1 described in Table 2 asserts that there exists a state ket $|\psi>$ that represents/describes the system. Next, corresponding to the 'amount of interest' in the above analogy, here in Table 2 we have postulate # 2 that says that there exists an operator $\hat{O}$ associated with a measurable O. Further, just like the method of determination of the amount of interest, postulate #3 described in the table provides the recipe of computing the average value of O from the knowledge of the state of the system $|\psi>$ and the operator $\hat{O}$. According to this postulate, the average value of O is:

$$< O> = < \psi|\hat{O}|\psi> .$$

For the meaning of the matrix element $< \psi|\hat{O}|\psi>$, Eq.(EC-1) and Table 5 in Appendix A would be helpful. To provide confidence and clarity, its application at many places has been illustrated in this section. While going through applications, the reader would note that the mathematical treatment discussed in this work does not need any complicated calculus or algebra or trigonometry. One needs to be familiar with only a few notations such as $|\psi>$, $< \psi |$, $< \psi| \phi>$, and $< \psi|\hat{O}|\phi>$. In this regard, the description of terms and notations of quantum mechanics given in Appendix A and B would be helpful.

A state may be denoted by different kinds of notations (see Appendix-A). In the text we have adopted Dirac's notations to denote the state ket. In Dirac's notation, a state is denoted by a label placed in the symbol | > (see Table 3). As an example of the state of a system, we again draw our attention to the problem of the two-stage gambling experiment. In Table 3, we have described various states. The first entry in this table gives a state |A> corresponding to accepting the second gamble, i.e., in this state (state of mind) the probability of accepting the second gamble is 100%. We denote this state by notation |A>. Here symbol | > is used to specify that it is a state ket (see Appendix –A). Before proceeding further, it may be appropriate for a reader to be familiar with all other states described in Table 3.

In Table 4, we describe two operators, $\hat{O}_W$ and $\hat{O}_A$. $\hat{O}_W$ operator corresponds to the probability of winning the first gamble and it may be named as 'win operator'. $\hat{O}_A$ operator corresponds to the probability of accepting the second gamble and it may be named as 'accept operator'.

Table 1: Notations regarding various probabilities

The events and activities are denoted as follows:

X_1: Win first gamble, X_2: Not win the first gamble,

A: Accept second gamble, B: Not accept second gamble

$p(X_1)$	Probability of winning the first gamble = No. of participants winning the first gamble/No. of participants.
$p(X_2)$	Probability of not winning the first gamble (losing) = No. of participants not winning the first gamble/No. of participants
$p(A\|X_1)$	Probability of accepting the second gamble after knowing that he/she has won the first gamble
$p(AX_1)$	Joint probability of winning the first gamble and accepting the second gamble. It is equal to the product of $p(X_1)$ and $p(A\|X_1)$. $p(AX_1) = p(X_1)\ p(A\|X_1)$. (1)
$p(A\|X_2)$	Probability of accepting the second gamble after knowing that he/she has not won the first gamble
$p(AX_2)$	Joint probability of not winning the first gamble and accepting the second gamble. It is equal to the product of $p(X_2)$ and $p(A\|X_2)$. $p(AX_2) = p(X_2)\ p(A\|X_2)$. (2)
$p(A)$	Probability of accepting the second gamble in absence of any knowledge of winning or losing the first gamble
$p(B\|X_1)$	Probability of not accepting the second gamble after knowing that he/she has won the first gamble
$p(BX_1)$ $p(B\|X_2)$ $p(BX_2)$ $p(B)$	See the corresponding terms with A in this table. The difference between A and B symbols is in 'accepting' and 'not accepting'. In this regard, parallel to Eqs. (1) and (2), we shall have: $p(BX_1) = p(X_1)\ p(B\|X_1)$, and (3) $p(BX_2) = p(X_2)\ p(B\|X_2)$. (4) It can be easily understood that the sum of probability of accepting and not accepting is 1, i.e., $p(A) + p(B) = 1$. (5)

Table 2: Postulates of quantum mechanics and quantum decision model

	Quantum mechanics	**Quantum decision model**
(1) State corresponding to a system	The dynamical state of a system can be fully represented by a state ket $\|\psi\rangle$.	The state of a system can be represented by a state ket $\|\psi\rangle$.
(2) Operator corresponding to a measurable	With every physical quantity (dynamical variable) O, an operator $\hat{O}$ can be associated.	With every prospect or a variable O an operator $\hat{O}$ can be associated.
(3) Result of measurement	The average result of measurement $\langle O\rangle$, of O in state $\|\psi\rangle$ is given by $\langle O\rangle = \langle\psi\|\hat{O}\|\psi\rangle/\langle\psi\|\psi\rangle$. When $\langle\psi\|\psi\rangle$ is normalized, i.e. for $\langle\psi\|\psi\rangle =1$, above relation becomes $\langle O\rangle = \langle\psi\|\hat{O}\|\psi\rangle$. (6)	The average result of measurement $\langle O\rangle$ of O in state $\|\psi\rangle$ is given by $\langle O\rangle = \langle\psi\|\hat{O}\|\psi\rangle/\langle\psi\|\psi\rangle$. When $\langle\psi\|\psi\rangle$ is normalized, i.e. for $\langle\psi\|\psi\rangle =1$, above relation becomes $\langle O\rangle = \langle\psi\|\hat{O}\|\psi\rangle$. (6a)

Table 3: Notations regarding quantum states

For mathematical simplicity all states described in this table are taken as normalized.

$\|A\rangle$	Represents a state corresponding to accepting the second gamble. In this state the probability of accepting the second gamble is 100%.
$\|B\rangle$	Represents a state corresponding to not accepting the second gamble. In this state the probability of not accepting the second gamble is 100%.
$\|X_1\rangle$	Represents a state in which the probability of winning the first gamble is 100%.
$\|X_2\rangle$	Represents a state in which the probability of winning the first gamble is 0% .
$\|AX_1\rangle$	Represents a state in which the probability of winning the first gamble is 100% as well as the probability of accepting the second gamble is 100%. It is the tensor product (see Appendix-B) of $\|A\rangle$ and $\|X_1\rangle$, i.e. $\|AX_1\rangle = \|A\rangle \|X_1\rangle$.
$\|AX_2\rangle$	Represents a state in which the probability of losing the first gamble is 100 % as well as the probability of accepting the second gamble is 100%. It is the tensor product of $\|A\rangle$ and $\|X_2\rangle$, i.e. $\|AX_2\rangle = \|A\rangle \|X_2\rangle$.
$\|BX_1\rangle$ $\|BX_2\rangle$	See the corresponding terms with A in this table. The difference between A and B symbols is in 'accepting' and 'not accepting'.

Table 4: Notations regarding operators

$\hat{O}_W$ (Win operator)	Operator corresponding to the probability of winning the first gamble. It has eigen values 1 and 0. Eigen value = 1 corresponds to the winning of the first gamble and 0 corresponds to losing the first gamble
$\hat{O}_A$ (Accept operator)	Operator corresponding to the probability of accepting the second gamble. It has eigen values 1 and 0. Eigen value = 1 corresponds to accepting the second gamble and 0 corresponds to not accepting the second gamble.

2.1 Experimental data to be explained

Before going further it would be appropriate to be familiar with, in our notations, the experimental data that we need to explain. The experiments performed by Tversky and Shafir (1992) reveal the following:

$$p(A|X_1) = 0.69, \qquad p(B|X_1) = 1 - p(A|X_1) = 0.31, \tag{7}$$

$$p(A|X_2) = 0.59, \qquad p(B|X_2) = 1 - p(A|X_2) = 0.41, \tag{8}$$

$$p(A) = 0.36, \quad \text{and} \quad p(B) = 1 - p(A) = 0.64. \tag{9}$$

The interesting part of these data is as follows: 69% participants are ready to accept the second gamble if they know that they have won the first gamble, and 59% participants are ready to accept the second gamble even if they know that they have lost the first gamble. But when they do not know the result of the first gamble, only 36% participants are ready to accept the second gamble. We here note that a majority of participants prefer to accept the second gamble in either case of win or lose but when they are uncertain about winning or losing, then only a small fraction of the participants are ready to accept the second gamble. In literature (Tversky and Shafir 1992), this is known as disjunction effect in choice under uncertainty.

Above data contradicts the sure-thing principle given by Savage (1954). According to this principle, if a prospect x is preferred to y knowing that event R happens, and if x is preferred to y with the knowledge that event R did not happen, then x should be preferred to y even when the result of happening of R is unknown. This principle is considered as one of the basic axioms of the rational classical theory of decision under uncertainty.

2.2 Eigenkets and eigenvalues

Using the hermitean nature of the operators in quantum mechanics, in general, it can be shown that if in state $|\phi_i\rangle$ the result of measurement of O is a_i with 100% certainty then following relation holds good:

$$\hat{O}\,|\phi_i\rangle = a_i\,|\phi_i\rangle \quad .$$

Here $\hat{O}$ is an operator associated with O. In quantum mechanics, we call $|\phi_i\rangle$ satisfying above relation as eigenket of operator $\hat{O}$ having eigenvalue a_i (see Appendix-A for more details). The reverse is also true, i.e., if the state of the system is an eigenstate of an operator $\hat{O}$ then the result of measurement of O in that eigenstate would be the corresponding eigenvalue and the uncertainty or variance in that result of measurement would be zero.

In the quantum decision model we assume that kets ($|\psi\rangle$), bras ($\langle\psi|$), and operators belong to the Hilbert space in the same way as we consider in quantum mechanics. Therefore, we get the same concept of eigenkets and eigenvalues in the quantum decision model as well. The scalar product $\langle\psi|\psi\rangle$ and matrix elements $\langle\psi\hat{O}\,|\psi\rangle$ are also defined in the same way as are in quantum mechanics (see Appendix-A).

In view of this description, we can say that state $|A\rangle$ is an eigenstate of operator $\hat{O}_A$ corresponding to eigen value 1, and state $|B\rangle$ is an eigenstate of the same operator corresponding to eigenvalue zero. Similarly, $|X_1\rangle$ is an eigenstate of operator $\hat{O}_W$ corresponding to eigenvalue 1 and $|X_2\rangle$ is an eigenstate of operator $\hat{O}_W$ corresponding to eigenvalue zero. In equation form we can write:

$$\hat{O}_A\,|A\rangle = |A\rangle\ , \qquad \hat{O}_A\,|B\rangle = \text{zero}|B\rangle = 0\ , \tag{10}$$

$$\hat{O}_W\,|X_1\rangle = |X_1\rangle\ , \qquad \hat{O}_W\,|X_2\rangle = \text{zero}\,|X_2\rangle = 0\ , \tag{11}$$

$$\hat{O}_A\,|AX_1\rangle = |AX_1\rangle\ , \quad \text{and} \quad \hat{O}_W\,|AX_1\rangle = |AX_1\rangle \quad . \tag{12}$$

For other similar equations, one may refer to Appendix – C. Further, we know that eigenkets belonging to different eigenvalues of a given operator are orthogonal (for details one may refer to Appendix –A). We use the same concept here also. Thus we shall have

$$\langle X_1|X_2\rangle = \langle X_2|X_1\rangle = 0 \quad , \qquad \langle A|B\rangle = \langle B|A\rangle = 0 \quad . \tag{13}$$

$$\langle AX_1|AX_2\rangle = \langle AX_2|AX_1\rangle = 0 \;, \quad \text{and} \quad \langle BX_1|BX_2\rangle = \langle BX_2|BX_1\rangle = 0 \;. \tag{14}$$

For more equations along these lines, please refer to Appendix – C. Here $\langle X_1|X_2\rangle$ is called the scalar product of kets $|X_1\rangle$ and $|X_2\rangle$; for details one may refer to Appendix –A.

2.3 Determination of the state of the system

We have operators and eigen states related with this experiment. All we need is to obtain the ket representing the state of the system. Let $|\phi\rangle$ gives the state of win/lose of the system. This state $|\phi\rangle$ in win-lose space (Hilbert space) would be a linear combination (for an illustration see Figure 1) of the related eigenket $|X_1\rangle$ corresponding to win and eigenket $|X_2\rangle$ corresponding to lose. Thus we can write

$$|\phi\rangle = \alpha_1|X_1\rangle + \alpha_2|X_2\rangle \quad . \tag{15}$$

For the sake of mathematical convenience, we choose the expansion coefficients α_1 and α_2 such that the ket $|\phi\rangle$ is normalized, i.e. $\langle\phi|\phi\rangle = 1$. This normalization condition would lead to

$$|\alpha_1|^2 + |\alpha_2|^2 = 1 \;. \tag{16}$$

As $\hat{O}_W$ operator corresponds to the probability of winning the first gamble, for computing the probability of winning the first gamble, we shall employ Eq.(6a) and this operator $\hat{O}_W$. Using Eqs.(6a), for the probability of winning the first gamble, we can write

$$p(X_1) = \langle\phi|\hat{O}_W|\phi\rangle \quad . \tag{17}$$

In view of Eqs.(15) and (11) we can get

$$\hat{\mathbf{O}}_{\mathbf{W}}|\phi\rangle = \alpha_1|X_1\rangle \; . \tag{18}$$

For the bra vector $\langle\phi|$, Eq.(15) leads to

$$\langle\phi| = \alpha_1^*\langle X_1| + \alpha_2^*\langle X_2| \quad . \qquad \text{(See Table 5 in Appendix A)} \tag{19}$$

Here superscript * has been used to indicate the complex conjugate, i.e., α_1^* is complex conjugate of the number α_1. By combining Eqs.(17)-(19), we get

$$p(X_1) = \langle\phi|\hat{\mathbf{O}}_{\mathbf{W}}|\phi\rangle = [\alpha_1^*\langle X_1| + \alpha_2^*\langle X_2|]\,[\alpha_1|X_1\rangle]$$

$$= [\alpha_1^*\alpha_1\langle X_1|X_1\rangle + \alpha_2^*\alpha_1\langle X_2|X_1\rangle] \; . \tag{20}$$

Since $|X_1\rangle$ is normalized, therefore, $\langle X_1|X_1\rangle = 1$. Eq. (20) in combination with this normalization condition and Eq. (13) leads to

$$p(X_1) = \alpha_1^*\alpha_1 = |\alpha_1|^2 \quad . \tag{21}$$

From this result, one can obtain

$$p(X_2) = 1 - p(X_1) = |\alpha_2|^2 \quad . \tag{22}$$

It is obvious that the win/lose is not a matter of decision by the players in this experiment. Further, we know that among the participants who have won the first gamble, some would accept the second gamble and some would not. As $|AX_1\rangle$ corresponds to winning the first gamble and accepting the second gamble, and $|BX_1\rangle$ corresponds to winning the first gamble and rejecting the second gamble, therefore, the state of mind $|\psi_1\rangle$ associated with the winning of the first gamble can be expressed (for an illustration see Figure 2) as a linear combination of $|AX_1\rangle$ and $|BX_1\rangle$:

$$|\psi_1\rangle = a_1|AX_1\rangle + b_1|BX_1\rangle \quad . \tag{23}$$

The normalization condition requires $\langle\psi_1|\psi_1\rangle = 1$. This would lead to

$$| a_1|^2 +| b_1|^2 = 1 \quad . \tag{24}$$

For computing the probability of accepting the second gamble with the knowledge of winning the first gamble, we can use Eq.(6a), operator $\hat{\mathbf{O}}_{\mathbf{A}}$ and state $|\psi_1\rangle$. Thus we have

$$p(A|X_1) = \langle\psi_1|\hat{\mathbf{O}}_{\mathbf{A}}|\psi_1\rangle \quad . \tag{25}$$

Above equation with the help of Eqs.(23), (12), (EC-11), (Ec-22), and the normalization property [$\langle AX_1|AX_1\rangle = 1$] leads to

$$p(A|X_1) = \langle\psi_1|\hat{\mathbf{O}}_{\mathbf{A}}|\psi_1\rangle = | a_1|^2 \quad . \tag{26}$$

As sum of $p(A|X_1)$ and $p(B|X_1)$ equals one, using Eq.(24) and (26) one can obtain

$$p(B|X_1\rangle = 1 - p(A|X_1) = | b_1|^2 .$$

Similarly, for the state of mind $|\psi_2\rangle$ (for an illustration see Figure 3) associated with losing the first gamble, one can write

$$|\psi_2\rangle = a_2|AX_2\rangle + b_2 |BX_2\rangle \ , \text{ where } | a_2|^2 +| b_2|^2 = 1 \quad . \tag{27}$$

Again, using operator $\hat{\mathbf{O}}_{\mathbf{A}}$, Eqs. (6a), (27), (EC-13), (EC-15), (EC-25), and the normalization property [$\langle AX_2|AX_2\rangle = 1$] we can get

$$p(A|X_2\rangle = \langle\psi_2|\hat{\mathbf{O}}_{\mathbf{A}}|\psi_2\rangle = | a_2|^2 \quad . \tag{28}$$

Further, since sum of $p(A|X_2)$ and $p(B|X_2)$ equals one, therefore, using Eq.(27) and (28) one can have

$$p(B|X_2\rangle = 1 - p(A|X_2\rangle = | b_2|^2 \quad .$$

To arrive at the state of mind $|\psi>$ representing all subjects (sum of those who are losing and those who are winning the first play), we would have linear combination of states $|\psi_1>$ and $|\psi_2>$. The expansion coefficients for this combination must be the same as occurring in Eq.(15), i.e.

$$|\psi> = \alpha_1|\psi_1> + \alpha_2 |\psi_2> \quad . \tag{29}$$

This equation implies that the state $|X_1>$ of Eq.(15) in win-lose space becomes state $|\psi_1>$ in win-lose-accept-reject space, the state $|X_2>$ of Eq.(15) in win-lose space becomes state $|\psi_2>$ in win-lose-accept-reject space and the state $|\phi>$ of the same equation in win-lose space becomes state $|\psi>$ in win-lose-accept-reject space. On substituting the value of $|\psi_1>$ given by Eq.(23) and of $|\psi_2>$ given by Eq.(27) into Eq.(29), we obtain

$$|\psi> = c_1|AX_1> + c_2|AX_2> + c_3 |BX_1> + c_4 |BX_2> \quad , \tag{30}$$

where $c_1 = \alpha_1 a_1$, $c_2 = \alpha_2 a_2$, $c_3 = \alpha_1 b_1$, and $c_4 = \alpha_2 b_2$. (31)

It may be noted that $|AX_1>$ represents a state corresponding to accepting the second gamble and winning the first gamble, $|AX_2>$ represents a state corresponding to accepting the second gamble and not winning the first gamble, $|BX_1>$ represents a state corresponding to not accepting the second gamble but winning the first gamble, and $|BX_2>$ represents a state corresponding to not accepting the second gamble and not winning the first gamble (for an illustration see Figure 4). These are four orthonormal eigenkets in four-dimensional win-lose-accept-reject space.

2.4 Analysis of Probabilities

For the joint probability of winning the first gamble and accepting the second gamble in the state given by Eq.(30), we use Eq.(6a), Eq.(30) and the operator as a product of two operators $\hat{\mathbf{O}}_\mathbf{A} \hat{\mathbf{O}}_\mathbf{W}$. Thus we shall have,

$$p(AX_1) = <\psi|\hat{\mathbf{O}}_\mathbf{A} \hat{\mathbf{O}}_\mathbf{W} |\psi> \quad . \tag{32}$$

Using Eqs. (30), (12), (EC-12), (EC-14), and (EC-16) we can write

$$\hat{\mathbf{O}}_{\mathbf{W}}|\psi> \; = c_1|AX_1> \; + c_3|BX_1> \; . \tag{33}$$

Operating this equation again by $\hat{\mathbf{O}}_{\mathbf{A}}$ and using Eqs.(12) and (EC-11) we have

$$\hat{\mathbf{O}}_{\mathbf{A}}\,\hat{\mathbf{O}}_{\mathbf{W}}|\psi> \; = \hat{\mathbf{O}}_{\mathbf{A}}\,[c_1|AX_1> \; + c_3|BX_1>] = c_1|AX_1> \; . \tag{34}$$

Using Eqs.(32), (34), (30), orthogonality relations given by Eqs. (14), (EC-22) and (EC-24), and the normalization condition [$<AX_1|AX_1> = 1$] we obtain

$$p(AX_1) = |\, c_1|^2 \; . \tag{35}$$

We can similarly have

$$p(AX_2) = |\, c_2|^2 \;, \quad p(BX_1) = |\, c_3|^2 \;, \quad \text{and} \quad p(BX_2) = |\, c_4|^2 \; . \tag{36}$$

Using Eqs.(35), (36), (31), (27), (24) and (16), it can be seen that

$$p(AX_1) + p(AX_2) + p(BX_1) + p(BX_2) = |\, c_1|^2 + |\, c_2|^2 + |\, c_3|^2 + |\, c_4|^2 = 1 \; . \tag{37}$$

Eq.(35) in combination with Eq. (31) gives: $p(AX_1) = \; = |\,\alpha_1|^2\,|\,a_1|^2$. This equation with the help of Eqs.(26) and (21) leads to

$$p(AX_1) = p(X_1)\, p(A|X_1), \tag{38}$$

as expected [see Eq.(1)]. Similarly, Eqs.(36), (31) and other related equations can be used to obtain Eqs.(2)-(4). Such a verification of Eqs.(1)-(4) shows the consistencies of various probabilities computed in this subsection.

2.5 Analysis of the situation when outcome of the first game is unknown

Equation (30) represents a state that contains the information regarding winning and losing. The experiments of Tversky and Shafir (1992) performed with 98 subjects to determine $p(A|X_1)$ and $p(A|X_2)$ can be described by this state together with Eq.(15). Next, for the determination of probability of accepting the second gamble without any information regarding winning and losing the first gamble, p(A), they performed the experiment after 10 days. In the experiments, there is no magic wand to wash the information of winning and losing from the mind of a participant. Therefore, Tversky and Shafir (1992) used time to wash the memory. But in mathematics, one can wash the information regarding pass and fail contained in Eq.(30) by deleting X_1 and X_2 labels. Thus corresponding to the state of mind of another batch of participants considered by Tversky and Shafir (1992), we would have a state $|\psi'>$ that can be obtained by ignoring X_1 and X_2 in $|\psi>$ given by Eq.(30), i.e.,

$$|\psi'> = c'_1|A> + c'_2|A> + c'_3\,|B> + c'_4\,|B> = (c'_1 + c'_2)\,|A> + (c'_3 + c'_4)\,|B> . \qquad (39)$$

We would see that c'_i , (i=1-4) in this equation, are same as c_i occurring in Eq.(30) except that they may differ in the phase factor, i.e.,

$$|c'_1| = |c_1|, \quad |c'_2| = |c_2|, \quad |c'_3| = |c_3|, \text{ and } |c'_4| = |c_4| . \qquad (40)$$

It can easily be seen that if we compute the expectation value of operator $\hat{\mathbf{O}}_{\mathbf{A}}$ to get the probability of accepting the second gamble with the information regarding winning and losing using Eq.(30), we would have the absence of interference between the probability amplitude terms c_1 and c_2, i.e., we would simply get this probability as a sum of $p(AX_1)$ and $p(AX_2)$. But with the state given by Eq.(39), we shall see that one gets p(A) that equals to a sum of $p(AX_1)$, $p(AX_2)$, and an interference term. The existence of such an interference term is not possible in the classical framework. It is a special feature of the quantum framework.

The difference in the expectation values of operator $\hat{\mathbf{O}}_{\mathbf{A}}$ in the states given by Eqs.(30) and (39) can be compared by using an analogy of a double slit experiment of physics which is performed to study interference of electrons with and without a watch over electrons near the slits. We know that (see Feynmann et.al. 1966) interference pattern is not observed when the electrons are watched to know the slit through which they pass. But when we do not have such watch then we get the interference pattern.

We may also look at equations (30) and (39) from the reverse perspective. In the double slit experiment (see Feynmann *et.al*. 1966), if we want to watch the electrons then we need to put the detectors near slits. In the same way, here if we want to insert the knowledge of outcome of the first gamble in Eq.(39), then we need to put

X_1 and X_2 in Eq.(39) such that we get the values of $p(AX_1)$, $p(AX_2)$, $p(BX_1)$, and $p(BX_2)$ same as given by Eq.(30) [see Eqs.(35)-(36)]. This would be possible when the values of coefficients c_i and c'_i, (i=1-4) are such that their magnitudes are equal [see Eq.(40)].

We can now compute p(A) by using Eqs.(6a) and (35), and the operator $\hat{\mathbf{O}}_{\mathbf{A}}$:

$$p(A) = \langle\psi'|\hat{\mathbf{O}}_{\mathbf{A}}|\psi'\rangle \ .$$

Using above equation together with Eqs. (39), (10), and (13), and the normalization condition $\langle A|A\rangle = 1$, we obtain

$$p(A) = |c'_1 + c'_2|^2 = |c'_1|^2 + |c'_2|^2 + c'_1{}^* c'_2 + c'_2{}^* c'_1 \quad . \tag{41}$$

In view of Eqs. (35), (36), and (40) above equation leads to

$$p(A) = p(AX_1) + p(AX_2) + q_{int}(A), \tag{42}$$

where, $q_{int}(A) = c'_1{}^* c'_2 + c'_2{}^* c'_1$. (43)

As c'_1 and c'_2 can be complex numbers, in view of Eqs.(35), (36) and (40), we can express c'_1 and c'_2 in terms of their absolute values and the respective phase angles θ_1 and θ_2 as follows:

$$c'_1 = |c'_1| \exp(i\theta_1) = [p(AX_1)]^{1/2} \exp(i\theta_1), \quad \text{and} \tag{44}$$

$$c'_2 = |c'_2| \exp(i\theta_2) = [p(AX_2)]^{1/2} \exp(i\theta_2) \ . \tag{45}$$

With these expressions, Eq.(43) yields:

$$q_{int}(A) = 2[p(AX_1)\, p(AX_2)]^{1/2} \cos(\theta_2 - \theta_1) \quad . \tag{46}$$

[It may be noted that $i = (-1)^{1/2}$ and $\exp(i\theta) = \cos(\theta) + i \sin(\theta)$].

Since the minimum and maximum possible values of the cosine are -1 and $+1$, respectively, therefore, using Eq.(46) one can find that the interference term $q_{int}(A)$ satisfies the following relation:

$$-2[p(AX_1)\,p(AX_2)]^{1/2} \le q_{int}(A) \le 2[p(AX_1)\,p(AX_2)]^{1/2} \quad . \tag{47}$$

Similarly, using Eq.(6a), Eq.(39), and the operator $[\hat{\mathbf{I}} - \hat{\mathbf{O}}_{\mathbf{A}}]$ for the probability of not accepting the second gamble, (here $\hat{\mathbf{I}}$ is a unit operator), one can get

$$p(B) = p(BX_1) + p(BX_2) + q_{int}(B) \quad , \tag{48}$$

where, $\quad q_{int}(B) = c'_3{}^{*}\, c'_4 + c'_4{}^{*}\, c'_3$. (49)

By writing equations similar to (44) and (45) for c'_3 and c'_4, we can get

$$q_{int}(B) = 2[p(BX_1)\,p(BX_2)]^{1/2} \cos(\theta_4 - \theta_3) \quad . \tag{50}$$

Combining Eqs.(42) and (48), we obtain

$$p(A) + p(B) = p(AX_1) + p(AX_2) + p(BX_1) + p(BX_2) + q_{int}(A) + q_{int}(B) \, . \tag{51}$$

Eq.(51) in association with Eqs.(37) and (5) gives

$$q_{int}(A) + q_{int}(B) = 0 \quad . \tag{52}$$

It may be noted that the main results of this treatment, Eqs.(42), (46), (48) and (50) have also been derived by Yukalov and Sornette (2009a) using the postulates and states very similar to that described here. The treatment presented here mainly differs from their treatment in the consideration of different kind of operators.

3. Discussion

3.1 Explanation of the two-stage gambling experiment

For the coin tossing experiment performed by Tversky and Shafir (1992), one can take $p(X_1) = 0.5$, and $p(X_2) = 0.5$. Further, using Eqs.(1), and (2), and the data given in Eqs.(7) and (8), we can obtain $p(AX_1) = 0.345$, and $p(AX_2) = 0.295$.

Classically, one expects the value of p(A) as sum of $p(AX_1)$ and $p(AX_2)$ (= 0.64). Against this expectation, we here get p(A) = 0.36 [see Eq.(9)]. Eq.(42), however, shows that this anomalous behavior can be explained by the value of $q_{int}(A)$ equal to – 0.28. This value of $q_{int}(A) = -0.28$ is consistent with Eq.(47) which gives

$$-0.638 \leq q_{int}(A) \leq 0.638 . \quad (53)$$

This consistency suggests that this treatment based on the interference occurring in the quantum decision model, though cannot predict in advance the results of the experiment, but can explain the results. It may be noted that these experimental results violate the classical axiom known as Savage's sure-thing principle (1954). Further, if we consider $\cos(\theta_2 - \theta_1)$ of Eq. (46) as an adjustable parameter, then we can say that the experimental data can be explained by assigning

$$\cos(\theta_2 - \theta_1) = -0.439. \quad (54)$$

This value of $\cos(\theta_2 - \theta_1)$ with Eq.(46) leads to $q_{int}(A) = -0.28$ that can explain the experimental results for accepting the gamble. Similarly, for the probability of rejecting to play the second gamble, we can get following results:

$$p(BX_1) = 0.155, \quad p(BX_2) = 0.205 , \quad p(B) = 0.64 , \quad (55)$$

$$q_{int}(B) = 0.28, \quad \text{and} \quad \cos(\theta_2 - \theta_1) = 0.785. \quad (56)$$

Yukalov and Sornette (2009a) argue that, under uncertainty(the lack of knowledge of the outcome of the first game), the decision in favor of an 'action' (accepting to play the second game) is more difficult than that in favor of an'inaction' (not to play). Therefore, with this assumption, out of two terms, $q_{int}(A)$ and $q_{int}(B)$, which add to 0 [see Eq.(52)], we can say so much in advance that $q_{int}(A)$ would be negative and $q_{int}(B)$ would be positive. Can we think of any method of predicting the value of $q_{int}(A)$ or phase angles θ_2 and θ_1 in advance? The visualization of any experimental procedure to determine phase angles in advance without any knowledge of p(A) or $q_{int}(A)$ seems to be beyond the scope of our present understanding? Probably, there may be some link between the phase angles and the distribution of time taken by different subjects in making a particular decision. The problem, however, is very complex and may be a matter of future research. At present, we can only say that using quantum theory we are able to explain the results of the two-stage gambling experiment which could not be explained by any classical formulation such as sure-thing principle of Savage (1954) or classical Markov model studied by Pothos and Busemeyer (2010).

3.2 Explanation of the buy-or-not-to-buy experiment

In this experiment of Tversky and Shafir (1992), the subjects (undergraduate students at Stanford University) were asked to imagine that they have just taken a tough examination, and at the end of the fall quarter, an attractive Christmas vacation package to Hawaii at very low price is being offered to them. One group of 67 subjects (say group-1) was asked to imagine that they have passed the examination, another group of 67 (say group-2) was asked to imagine that they have failed the examination, and the third group of 66 (say group-3) was asked to imagine that the outcome of their examination is not known to them.

In this experiment, 54% subjects from group-1, 57% subjects from group-2, and 32% subjects from group-3 were ready to buy the vacation package. The analysis of these data has been presented and discussed in Appendix-D. From the analysis presented there, we note that these experimental results can be explained by relations similar to the interference equations derived and discussed in section (2.5). Thus again we see that using a quantum model we are able to explain the results of buy-or-not-to-buy experiment of Tversky and Shafir (1992) which could not be explained by any classical formulation such as sure-thing principle of Savage (1954).

3.3 Classical versus quantum model

In classical formulation, the interference term $q_{int}(A)$ given in Eq.(42) remains absent. According to the classical statistics, a simple addition of probabilities $p(AX_1)$ and $p(AX_2)$ equals $p(A)$. To explain the results of two-stage gambling experiment and other experiments, Pothos and Busemeyer (2010) have employed quantum as well as Markov (classical) models. While comparing different models, at one place they write: "*Although cognitive dissonance tendencies can be implemented in both the Markov and quantum models, we shall see that it does not help the Markov model, and only the quantum model explains the sure thing principle violations*."

The following essential differences between the classical and quantum formulations are worth noting:

(a) In a quantum model, the probabilities are expressed as the squares of the probability amplitudes c_1 , $c_{2, \ldots}$ [see Eqs.(35)-(36)]. In classical formulation, we do not have such terms as probability amplitudes.

(b) In a quantum model, for getting p(A) instead of adding the respective probabilities, the respective probability amplitudes get added, and then p(A) is obtained by squaring the probability amplitudes. Thus the probability amplitude for p(A) equals $(c'_1 + c'_2)$, and p(A) equals $|(c'_1 + c'_2)|^2$ [see Eq.(41)]. The interference term $q_{int}(A)$ automatically appears when we compute the value of $|(c'_1 + c'_2)|^2$. But in classical formulation, instead of addition of probability amplitudes the probabilities are added. Thus, classically, p(A) equals to sum of $p(AX_1)$ and $p(AX_2)$.

(c) The probability amplitudes, in general, may be complex numbers [see Eqs.(44)-(45)]. The phase factors affect the magnitude of the interference term, $q_{int}(A)$. In classical framework, we do not have such phase factors or complex numbers in connection with the probabilities. It may be added that the value of the interference term given by a quantum model may be zero also. In such case the quantum results merge to the classical results. This happens when the value of cosine term of Eq.(46) or (50) equals zero.

Without an explanation of the disjunction effect with the help of a quantum interference term, one may only say that the disjunction effect, as observed in the two-stage gambling experiment, may be due to lack of sound thinking of the subjects under uncertainty. In this regard, Tversky and Shafir (1992) write: " *We suggest that, in the presence of uncertainty, people are often reluctant to think through the implications of each outcome and, as a result, may violate STP (sure-thing principle)*."

3.4 Quantum decision theory of Yukalov and Sornette

Various features presented here are the same as given by the quantum decision theory (QDT) of Yukalov and Sornette (2009a). The main difference is in the selection of operators $\hat{O}$ associated with Eq.(6a). In section 2, we have considered the operators characterized by their explicit operations (such as 'win', 'accept') through their eigenkets and eigenvalues. In QDT, the operators are expressed in terms of prospect states. For example, they consider an operator $(|\alpha_{11}|^2 |AX_1\rangle \langle AX_1|)$ corresponding to the prospect state $(\alpha_{11}|AX_1\rangle)$. By prospect state they mean the state in which one is interested in reaching from the given state of mind. It may be noted that in physics also both kinds of operators [the operators associated with observables similar to what we have described here, and operators of kind |s> <s| (a ket multiplied by a bra on right) similar to what Yukalov and Sornette (2009a) have employed] are used.

It is interesting to note that the final results given by Eqs.(42), (46), (48), (50), (53), (54), (EC-33), and (EC-34) are in total agreement with those given by Yukalov and Sornette (2009a). As regards Eqs.(35)-(36) and (41), these results would also be in agreement with the QDT results provided their parameters have following values:

$$|\alpha_{ij}| = 1, \text{ for } (i = 1, 2, \text{ and } j=1, 2). \tag{57}$$

Here parameters α_{ij} corresponds to the expansion coefficients occurring in Eqs.(27) and (28) of Yukalov and Sornette (2009a) (for a 2 × 2 dimensional case; i =1 and 2 correspond to our A and B, and j =1 and 2 correspond to our X_1 and X_2 , respectively).

In absence of the knowledge of explicit four equations required for the determination of α_{11}, α_{12}, α_{21}, and α_{22}, it is not possible to know various possible values of these parameters of Yukalov and Sornette (2009a). However, it can be verified that the values of these parameters given by Eq.(57) do not disagree with the requirements described by Yukalov and Sornette (2009a).

In view of the fact that Eq.(57), which has been obtained by comparing the present results and those of QDT, does not disagree with the requirements described by Yukalov and Sornette (2009a), and as there are not sufficient number of equations to determine these parameters α_{ij}, the present work also becomes valuable to the formulation of Yukalov and Sornette (2009a) in the sense that it provides a way (if not 'the way') to determine α_{11}, α_{12}, α_{21}, and α_{22}.

In the present work, for determining the probability of accepting the second gamble when the result of the first gamble is not known, for the state of the mind of the participants we employ Eq.(39), not Eq.(30). In addition, we use Eq. (6a) and the accept operator $\hat{\mathbf{O}}_{\mathbf{A}}$ to compute the probability of accepting the second gamble. Thus we do not involve variables X_1 and X_2 related with win and lose in such determination. But Yukalov and Sornette (2009a) employ the state of mind given by Eq.(30) and an operator that includes variables X_1 and X_2. It is interesting to note that their final results are exactly same as obtained here [Eqs.(42) and (46)]. Such an agreement creates scope for further advancement in this area. A search for the cause of arriving at the same result by two different paths may be helpful in gaining a finer understanding of the paths.

3.5 Derivation of empirical results of Khrennikov

In an attempt to explain some experiments related with cognitive decision making and information processing, Khrennikov (2009) describes the necessity of a quantum-like model. He assumes the requirement of two types of interference: (1) the conventional trigonometric (cos-type) interference, and (2) the hyperbolic (cosh-type) interference. His cos-type interference model assumes the existence of equations like Eqs.(42) and (46) described in this work. In view of the derivation of these equations presented here, one can say that the interference equations, such as Eqs. (42) and (46), need not be assumed. These equations are simply based on the basic postulates (see Table 2) and the properties of states belonging to the Hilbert space.

3.6 Quantum model of Pothos and Busemeyer

The disjunction effect related with two-state gambling experiment of Tversky and Shafir (1992) has also been explained by Pothos and Busemeyer (2010), using an alternative quantum model and a few adjustable parameters. Their quantum probability model is based on an equation parallel to the time dependent Schrodinger equation of quantum mechanics (e.g. see Messiah1961). They employ this equation to study the effect of external information on the evolution of the state of the mind with time.

Besides the explanation of the results related to the psychology experiments, the concluding remarks of Pothos and Busemeyer (2010) regarding the quantum nature of human cognition are worth noting. They write: "*Finally, recent results in computer science have shown quantum computation to be fundamentally faster compared with classical computation, for certain problems (Nielsen & Chung 2000). Perhaps the success of human cognition can be partly explained by its use of quantum principles*."

3.7 Application to other decision problems

Above description reveals that in many situations the decisions taken by a human mind cannot be understood by a classical model but can be explained by a quantum model. The usefulness of the equations related with quantum mechanics in psychology suggests that quantum models may be useful to other disciplines also where the human psychology plays an important role. One can argue that a successful and valuable application of quantum models to the mainstream of business related decisions is not a matter of 'if' but of 'when', 'where', and 'how'. In this connection, the following remarks of Overman(1996) presented a while back are also worth noting: *" The experimentation and adoption of the metaphors and methods of chaos and quantum theory hold new promise for the management sciences in the next century. It is not so much that traditional social scientific methods have become obsolete; it is that we have a continuing need to expand the scope and power of our methods just to keep pace with our organizational realities."*

Group decision making in virtually any setting (e.g., DeSanctis and Gallupe 1987), collaborative or extended supply chain management (e. g., Guide and Van Wassenhove 2009), merger & acquisition of business firms (e.g., DePamphilis 2010) are some examples of the decision related problems of interest. Can quantum mechanics provide a new and valuable insight in such areas? The challenging task in tackling such problems is to find the operators and eigenkets corresponding to the key variables associated with the problem. It may be a long way to realize the potential of a quantum model and to arrive at a successful and valuable application of the quantum mechanical framework to the mainstream of business. However, at this stage, it may be worthwhile to see how some issues related with a business problem can be expressed in the notations of the quantum decision model.

With this objective, we consider an example of merger/acquisition of a firm (say, firm B) by another firm A. First, in addition to the financial interests, the psychology of managers and board members of firm A and B plays a major role in arriving at the merger deal. Next, immediately after the announcement of the merger deal, the role of the public perception influenced by the comments and analysis by experts regarding the synergy, ego, and other aspects related with the merger influences the share price of firms A and B. But here our interest is just to introduce the concept, therefore, we shall limit our discussion on the state of mind of the acquiring firm.

This example would serve an additional purpose. It would illustrate that a quantum decision model has a wider applicability. One can use it even when interference effect, similar to that described by Eqs.(42) and (46), does not take place.

3.7.1 Merger and acquisition problem: State of mind of the acquiring firm

There is a large body of literature in finance and strategy that has studied mergers & acquisitions (e.g., Malmendier and Tate 2008; Morellec and Zhdanov 2005; Shleifer and Vishny 2003; Tichy 2001; Andrade 2001). The purpose of this illustration here is only to show the potential application of quantum mechanics to analysis of this common phenomenon. Thus we take a simplified view of mergers & acquisition issues. Let us denote the state of mind of the acquiring firm A by $|\Psi_A\rangle$. Using the expansion postulate (see Appendix-A), it can be expressed as a linear combination of related eigenkets of the price operator:

$$|\Psi_A\rangle = a_1 \, | K_A\rangle + a_2 \, | O_A\rangle \quad . \tag{58}$$

Here a_1 and a_2 are expansion coefficients. The normalization condition [$\langle\Psi_A|\Psi_A\rangle = 1$] gives:

$$|a_1|^2 + |a_2|^2 = 1 \quad . \tag{59}$$

$| K_A\rangle$ denotes a component of the state of the mind of firm A related with the financial factors and public perception (as viewed by the firm A) responsible for the stock price of A. Here, the symbol K_A is chosen with an additional purpose. K_A in $| K_A\rangle$ is such that the market value of stock A, as perceived by firm A, after the announcement of the merger is K_A times the present market value (p_a) of stock A. The value of K_A is usually close to unity. The firm A is of the view that the merger would be welcomed by the public such that the net value of all shares of A and B would be $S^{(A)}$ times the current market value of the same, where $S^{(A)}$ is a synergy dependent factor. The synergy factor $S^{(A)}$ as perceived by A is absorbed in K_A. K_A may be time dependent but here we are confining our attention to the value of K_A within a short duration after the announcement of the merger.

$| O_A\rangle$ denotes a component of the state of the mind of firm A that may contain fear (fear of competitors acquiring B, in case this proposed merger of B with A does not take place), ego (ego of becoming a big firm), greed, and other factors, such as future synergy perceived by A but not currently perceived by the public, that do not contribute to the market price within a short duration after the announcement of the deal. The firm A may realize that the public would not be able to visualize some synergy factors in near future. According to A, such synergy factors would not contribute to the market value of A immediately after the announcement of the merger deal. The effect of such synergy factors are also absorbed in the coefficient a_2.

In Appendix-E, using the concept of a price operator $\hat{\mathbf{P}}_{\mathbf{A}}$ and related eigenkets and eigenvalues we have shown that the expected market value of the stock of A as perceived by A in the state of mind $|\Psi_A>$ is

$$P^A = <\Psi_A|\hat{\mathbf{P}}_{\mathbf{A}}|\Psi_A> = (1 - |a_2|^2)p_a K_A \quad . \tag{60}$$

This equation is very interesting. If $a_2 = 0$, then $|\Psi_A> = |K_A>$, and in this state of mind the firm A expects to see the price of their stock after the announcement of the merger to be $p_a K_A$. But the presence of the non-zero value of a_2 shows that firm A is ready to be satisfied if the market value of their stock is $(1-|a_2|^2)$ times $p_a K_A$. In other words, in view of other possibilities, such as fear of competitors, demand of ego to be a big firm, greed, and future synergy perceived by them but not perceived by the public, they are ready to sacrifice the price of one share (immediately after announcement of the merger deal) by an amount equal to $|a_2|^2 p_a K_A$ in favor of firm B. The value of $|a_2|$ may be unknown to them but at the negotiation table $|a_2|$ may evolve to the right size to match the state of mind of firm A with that of B, in case the merger deal is done. Good negotiators from both sides may try to alter the parameters of each others' state of mind to their favor. It may be noted that in some situations, P^A given by Eq.(60) may be less than p_a. For $K_A = 1$, it is certainly less than p_a when $|a_2|^2$ is non-zero.

The presence of a_2 term in Eq.(58) is not to be considered as a weakness of A. Its existence facilitates the merger. Even in chemistry, the attraction between positive and negative charges facilitates the bonding between two atoms. Due to this a_2 term, gain to B becomes larger than that to A, and the deal becomes possible. If the negotiating team members are chosen such that in their mind the value of a_2 is very small then the chances of concluding a deal could be low. On the other hand, if it is very large then the premium payable to B would be very large.

In addition to the advantage of synergy factor to B, firm B would have an additional gain due to the sacrifice made by A due to the presence of a_2 term. If the total number of shares of firm A is N_A then the sacrifice $|a_2|^2K_Ap_a$ per share by A would mean an additional gain equal to $|a_2|^2K_Ap_a(N_A/N_B)$ to one share of B. Since (N_A/N_B) is usually vary large as compared to 1, the gain per share may be very large to B even for a small value of a_2. It is interesting to note that the market value of stock of acquiring firm usually falls on the announcement of merger. A study conducted by Andrade et al. (2001) over 3688 completed mergers during 1973-1998 reveals that an acquiring firm on average loses 0.7% and the target firm gains 16% on announcement. The interesting point of this study is that these data are nearly same for the three decades of the study: 1973-1979, 1980-89, and 1990-98.

The simple model presented above illustrates the potential of applying a quantum mechanics framework to study phenomenon encountered in practice where the state of mind of the players can be taken into account. Of course, measurement of parameters such as a_2 remains a topic for further study.

4. Summary and concluding Remarks

The most successful theory of physics, quantum mechanics, has a wide range of applicability in physics, chemistry, biology, cosmology, etc. In recent years, its potential to quantum cryptography and computation is also being explored. In social sciences, its application to economics and psychology appears very promising. Recently observed success of quantum models (e.g. Pothos and Busemeyer 2010; Khrennikov 2009; Yukalov and Sornette 2009a) in explaining some experimental results of psychology, such as disjunction effect observed by Tversky and Shaffir (1992), which could not be explained by the conventional (classical) theories adds a quantum dimension to the human decision making process. In view of such success, one may argue for the need to explore use of quantum mechanics in other branches of knowledge such as business where the human decision making is involved.

With an objective of introducing various important and basic concepts and mathematical equations associated with any quantum treatment we here have described a quantum model with focus on the explanation of the disjunction effect experimentally observed by Tversky and Shaffir (1992). Due to basic differences between the physical objects and the human decision making processes, one expects to see some differences between the quantum mechanics as used in physical sciences and the quantum models applicable to the human decision making processes. With a minimum number of changes in the basic terms, postulates, and equations of quantum mechanics , here we have described a quantum decision model suitable for the decision making processes. Essentially, the model described here is based on the quantum decision theory developed by Yukalov and Sornette (2009a). The application of postulates analogous to those of quantum mechanics and the selection of state of mind are similar to that given by Yukalov and Sornette (2009a). But here instead of prospect operators, we have considered more general and simpler kind of operators (such as win operator, accept operator) to derive the same final results as derived by Yukalov and Sornette (2009a) so that the range of applicability may widen and it becomes easier to apply to the business related problems.

The description of basic postulates of quantum mechanics and corresponding postulates of the quantum decision model, in almost similar words, has been presented in Table 2. While introducing terms of quantum mechanics necessary for the quantum decision model, we have followed an approach of minimum necessary details in section 2 with additional information in Appendix A. Further, with an objective of ease in comprehension, we have introduced operators, eigenkets, eigenvalues, state of mind, etc. with practical examples and avoided a formal rigor associated with a general case. It is expected that with the understanding of the application of these terms and equations as described here, it would be easier to follow the related formal rigorous terminology of quantum mechanics as described in the text books of physics for more advanced applications.

To illustrate the success and to explain the various terms and equations of the quantum model, first it has been applied to two experiments performed by Tversky and Shafir (1992) related with the disjunction effect. As discussed earlier, the results of these experiments could not be explained by the conventional (classical) theories but have been explained in recent years by various quantum models (Pothos and Busemeyer 2010; Khrennikov 2009; Yukalov and Sornette 2009a).

The study of merger/acquisition of two business firms has been chosen to consider a business related decision problem. Various concepts of the model such as price operator ($\hat{\mathbf{P}}_{\mathbf{A}}$) and its eigen kets [$|K_A>$, $|O_A>$] with the corresponding eigen values [p_aK_A, and zero], the normalization and orthogonality conditions [Eqs.(EC-

37) and (EC-38)], expansion coefficients [a_1, a_2, and their relationship given in Eq.(59)], etc. related with this problem have been discussed. In this model, we also note the significance of the term $a_2|O_A>$ which leads to a decrease in the market value of the stock of the acquiring firm and an increase in the market value of the stock of the acquired firm immediately after the announcement of the merger deal. More investigations are required to incorporate the effect of the views of the board members, share holders, and public perception in the quantum exploration of the merger problem.

In addition to quantum interference to explain the disjunction effect, and quantum superposition of states to explain the merger problem, quantum mechanics provides many special features such as Heisenberg's uncertainty principle, quantum tunneling, quantum theory of measurement, etc. (e.g., Razavy 2003; Messiah 1961; Stenholm and Suominen 2005; Bohm 1959), which can accommodate various kinds of diversities associated with human decisions, in general, and business related decisions, in particular. Further investigations on the quantum models of information exchange and market psychology (Choustova 2007; Haven 2008), quantum finance (Schaden 2003, Baaquie 2009a, Bagarello 2009), application of quantum mechanics to human dynamics related with health care management (Porter-O'Grady, 2007), quantum administration (Overman, 1996), quantum probabilistic behavior (Bordley1997) , etc. are needed to make a quantum difference in the world in this 21^{st} century.

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Figures

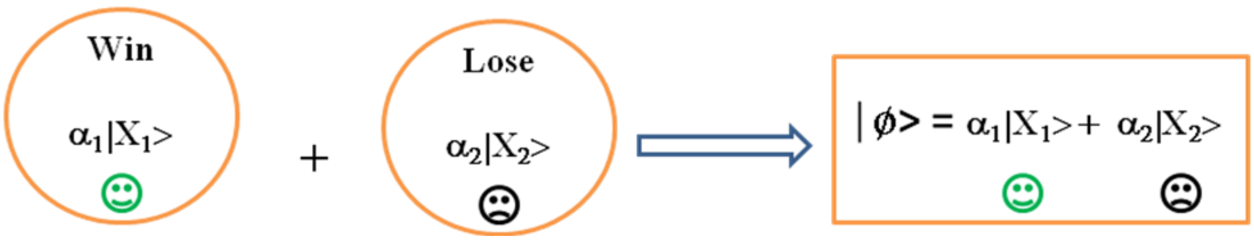

Figure 1: The win-lose state $|\phi>$ of the first game is a linear combination of win state ($|X_1>$) ('smiling face') with probability amplitude α_1, and the lose state ($|X2>$) ('sad face') with probability amplitude α_2. [See Eq.(15)].

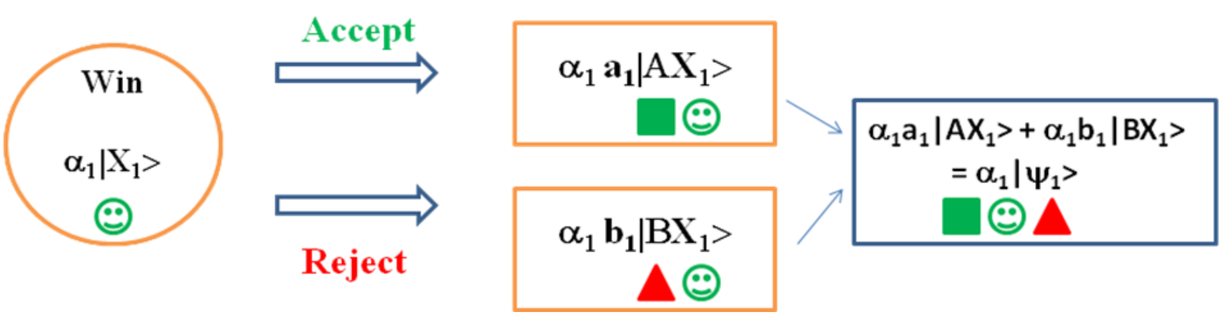

Figure 2: Win state leads to win and accept state (rectangle and 'smiling face' with probability amplitude a_1, and reject-win state (triangle and 'smiling face') with probability amplitude b_1. The linear combination of these two states has been expressed as $|\psi_1>$ [See Eq.(23)]. The factor α_1 in every block reminds the probability amplitude of the win state (see Figure 1).

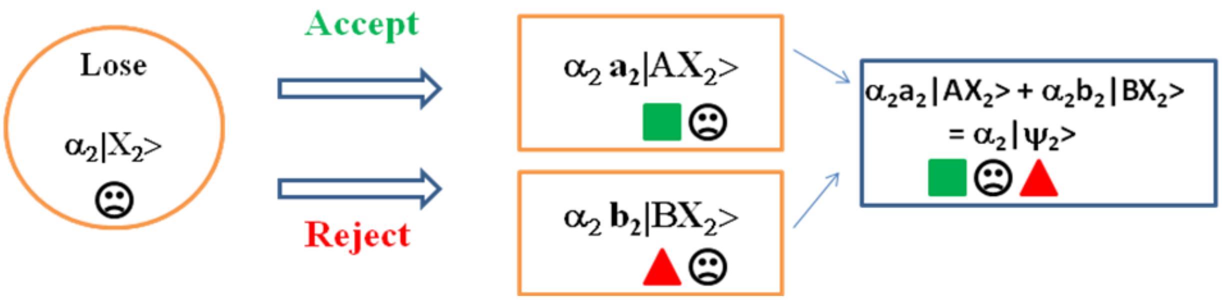

Figure 3: Lose state leads to accept and lose state (rectangle and 'sad face') with probability amplitude a_2, and reject-lose state (triangle and 'sad face') with probability amplitude b_2. The linear combination of these two states has been expressed as $|\psi_2>$ [See Eq.(27)]. The factor α_2 in every block reminds the probability amplitude of the lose state (see Figure 1).

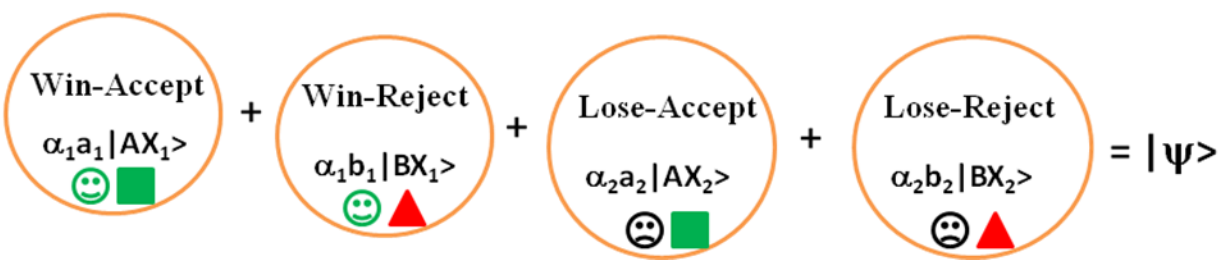

Figure 4: The state of the system $|\psi>$ given by Eq.(30) is a linear combination of four states corresponding to win-accept , win-reject, lose-accept, and lose-reject states.

Appendix- A. Basic terms of quantum mechanics

A-1: Wave functions

A function which is capable of representing a given system (quantum system) is known as a wave function. It is denoted by symbols such as ψ, ϕ, etc. It is a function of coordinates of configuration space, q_1, q_2, …, q_R, associated with the system.

The wave functions of quantum mechanics belong to a special function space known as Hilbert space. The following points are to be noted in connection with the concept of wave functions of Hilbert space (e.g., see Messiah 1961; von Neumann1983).

(i) Square-integrable:

The wave functions are square integrable functions. By square-integrable function, we mean a function that satisfies the following criteria:

$$\int \psi^* \psi \, d\tau \quad \text{converges.}$$

Here $d\tau = dq_1 dq_2 \ldots dq_R$, ψ^* is the complex conjugate of ψ, and the limit of integration extends over the whole space.

(ii) Linear space:

If ψ_1 and ψ_2 are the functions of the Hilbert space then a function

$$\psi = \lambda_1 \psi_1 + \lambda_2 \psi_2 \quad ,$$

also belongs to this space, where λ_1 and λ_2 are arbitrarily chosen numbers that may be complex.

(iii) Scalar product:

The scalar product of a function ψ by another function ϕ is denoted as $\langle \phi, \psi \rangle$ and is defined as

$$\langle \phi, \psi \rangle = \int \phi^* \psi \, d\tau \quad . \qquad \text{(EC-1)}$$

(iv) Orthogonal wave functions

If the scalar product of two functions is zero,

$$\langle \phi, \psi \rangle = 0, \qquad \text{(EC-2)}$$

then the functions ψ and ϕ are known as orthogonal to each other.

(v) Norm of a function and the normalized function

The norm N of a function is the scalar product of the function with itself. It is real and non-negative number.

$$N = \langle \psi, \psi \rangle \quad .$$

If N =1, then the function is called the normalized wave function.

(vi) Other useful relations:

$$\langle \lambda_1\phi_1 + \lambda_2\phi_2 \, , \psi \rangle = \lambda_1^* \langle \phi_1 , \psi \rangle + \lambda_2^* \langle \phi_2 , \psi \rangle \quad .$$

$$\langle \phi \, , \lambda_1\psi_1 + \lambda_2\psi_2 \rangle = \lambda_1 \langle \phi , \psi_1 \rangle + \lambda_2 \langle \phi , \psi_2 \rangle \quad .$$

Here λ_1 and λ_2 are constants.

(vii) State function: A wave function that defines a given quantum mechanical system is known as the state function of the given system.

A-1.1. Operators

If by some rule (prescription), from one wave function ψ we obtain another wave function ϕ, then we can express that rule in the mathematical form with the help of a term called operator $\hat{\mathbf{O}}$, as follows:

$$\phi = \hat{\mathbf{O}}\,\psi \quad .$$

Example: If the rule to obtain ϕ from ψ is to differentiate the function ψ with respect to x, then the operator here would be a differential operator, $\hat{\mathbf{O}} \equiv d/dx$; and, if

$$\psi = \exp(ax), \quad \text{then}$$

$$\phi = \hat{\mathbf{O}}\,\psi = a \exp(ax) \quad .$$

It may be noted that the operators associated with quantum mechanics are linear. **A linear operator** $\hat{\mathbf{O}}$ satisfies the following relation:

$$\hat{\mathbf{O}}(\lambda_1\psi_1 + \lambda_2\psi_2) = \lambda_1 \hat{\mathbf{O}}(\psi_1) + \lambda_2 \hat{\mathbf{O}}(\psi_2) \quad .$$

where λ_1 and λ_2 are arbitrarily chosen constants that may be complex.

The following relations in connections with the linear operators are also worth noting:

$$(c\,\hat{\mathbf{O}})\,\psi = c\,(\hat{\mathbf{O}}\,\psi) ,$$

$$(\hat{\mathbf{A}} + \hat{\mathbf{B}})\,\psi = (\hat{\mathbf{A}}\,\psi) + (\hat{\mathbf{B}}\,\psi) ,$$

$$(\hat{\mathbf{A}}\,\hat{\mathbf{B}})\,\psi = \hat{\mathbf{A}}\,(\hat{\mathbf{B}}\,\psi) \quad .$$

Here c is a constant, and $\hat{\mathbf{O}}$, $\hat{\mathbf{A}}$, and $\hat{\mathbf{B}}$ are linear operators.

It may also be noted that $(\hat{\mathbf{A}}\,\hat{\mathbf{B}})\,\psi$ may or may not be equal to $(\hat{\mathbf{B}}\,\hat{\mathbf{A}})\,\psi$. In general, they are unequal. If they are equal for all values of ψ, then we say that the operators $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ commute with each other.

The operators associated with quantum mechanics are also hermitean. **A hermitean operator** $\hat{\mathbf{O}}$ satisfies the following relation:

$$\int \phi^*(\hat{O}\,\psi)\,d\tau \;=\; \int (\hat{O}\,\phi)^*\psi\,d\tau\,. \qquad \text{(EC-3)}$$

Here ϕ and ψ are arbitrarily chosen wave functions.

A-1.2. Eigenfunctions, eigenvalues and expansion postulate

If a wave function ϕ_i satisfies the following relation:

$$\hat{O}\,\phi_i = a_i\,\phi_i\ , \qquad \text{(EC-4)}$$

where a_i is a constant, then ϕ_i is called an eigenfunction of the operator $\hat{O}$ and a_i is called the eigenvalue of the operator $\hat{O}$ corresponding to the eigenfunction $\phi_{i.}$ The following points are worth noting in this regard:

(i) More than one eigenfunction:

There can be n (n = two or more than two) linearly independent eigenfunctions of a given operator, i.e.,

$$\hat{O}\,\phi_1 = a_1\,\phi_1\ ,\ \hat{O}\,\phi_2 = a_2\,\phi_2\ ,\ \ldots,\ \hat{O}\,\phi_n = a_n\,\phi_n\ .$$

A set of n functions are said to be linearly independent, if none of these functions can be expressed as a linear combination of the remaining (n-1) functions.

It can be proved that eigenfunctions belonging to different eigenvalues are orthogonal, and a set of linearly independent eigenfunctions belonging to the same eigenvalue can be constructed such that they are also orthogonal. (eigenfunctions belonging to the same eigenvalue are called degenerate eigenfunctions.)

There may be a continuous set of eigenvalues and eigenfunctions also. For details one may refer to Messiah (1961).

(ii) Orthonormal set of n eigenfunctions

It is possible to have a set of n linearly independent eigenfunctions ($\phi_1, \phi_{2, \ldots,} \phi_n$) of an operator $\hat{\mathbf{O}}$ such that each function is normalized and different eigenfunctions are orthogonal to each other, i.e., the scalar product satisfies the following relation:

$$< \phi_i , \phi_j > = \delta_{ij} \quad . \qquad \text{(EC-5)}$$

Here δ_{ij} is known as the Kronecker delta. It satisfies the following properties.

$$\delta_{ij} = 1, \qquad \text{for } i = j,$$

and $\delta_{ij} = 0, \qquad \text{for } i \neq j \quad .$

(iii) Expansion postulate:

Any arbitrary wave function ψ can be expressed as a linear combination of a complete orthonormal set of n linearly independent eigenfunctions ϕ_i (i = 1 to n) of a hermitean operator as follows:

$$\psi = \sum_{\mathbf{i}} (c_i\, \phi_i) , \quad (i = 1 \text{ to } n) , \qquad \text{(EC-6)}$$

where

$$c_i = \int \phi_i^{*} \psi \, d\tau \quad .$$

It may be noted that Eq.(EC-6) also becomes useful in the determination of ψ if c_i and ϕ_i for all possible values of i are known.

A-2: Dirac's ket and bra vectors

An alternative approach to describe the formulation of quantum mechanics is to use the notations and concept of ket vectors as given by Dirac (e.g., see Messiah 1961; Dirac 1958). These ket vectors also constitute a linear vector space and satisfy the requirements of Hilbert space. A comparison of wave function notations and Dirac's notations is given in Table 5.

Table 5: A comparison of wave function and Dirac notations of quantum mechanics

Wave function notations	**Dirac's notations**
ψ	$\| \psi>$
ψ^*	$< \psi\|$
$< \phi, \psi>$	$<\phi\|\psi>$ (Scalar product)
$\int \phi^*(\hat{\mathbf{O}}\, \psi)\, d\tau$	$<\phi\|\hat{\mathbf{O}}\, \|\psi>$ (Matrix element)
$\hat{\mathbf{O}}\ \phi_i = a_i\, \phi_i$	$\hat{\mathbf{O}}\ \| \phi_i > = a_i \| \phi_i >$
$< \phi_i , \phi_j > = \delta_{ij}$.	$< \phi_i \|\phi_j > = \delta_{ij}$.
$\psi = \sum_{\mathbf{i}} (c_i\, \phi_i)$, (i = 1 to n)	$\|\psi> = \sum_{\mathbf{i}} (c_i \|\phi_i >)$, (i = 1 to n)

It may be noted that a label in association with the symbol | > is called ket or ket vector and a label in association with symbol < | is called bra or bra vector. ('*bra*' and '*ket*' have been derived from the word '*bra*c*ket*'). $<\phi|\hat{\mathbf{O}}\, |\psi>$ is sometimes called as a matrix element.

It is not necessary to use $| \phi_i >$ corresponding to the function ϕ_i. One can even use | i > or any other symbol just sufficient to distinguish it from others. In this vector space formulation of quantum mechanics, it is also not necessary to perform integrations as indicated in Eq. (EC-1). The quantum mechanics formulated in this way does not require explicit values of operators and state functions. From the knowledge of eignevalues and behavior of eigenkets of an operator, one can determine the values of various scalar products. These scalar products together with the postulates of quantum mechanics can be used to arrive at many useful results.

The ability of having the scalar product without performing integration, and the possibility of dealing with the states and operators without knowing their explicit values make the formulation of quantum mechanics applicable to other branches of knowledge. The present study of employing the mathematical framework of quantum mechanics illustrates this point. One notes that at many places in the text, we have computed scalar products from the knowledge of the orthonormal property of the related eigenkets [For example, in the text, we assigned orthonormal properties given by Eqs. (13) and (EC-17), and used these properties in arriving at Eq.(21) from Eq.(20)].

Appendix- B. Tensor product of two vector spaces

In the present study, in the first example, we considered win-lose space (two dimensional), accept-reject space (two dimensional), and their product space named as win-lose-accept-reject space (4 dimensional). Similarly, in the next example, we discussed pass-fail space, buy-not-buy space, and their product space named as pass-fail-buy-not-buy space.

Such a consideration of the product space has been done in accordance with the provisions of the tensor product of two vector spaces as employed in quantum mechanics (e.g., Messiah 1961). A vector space S_1 (N_1 dimensional) and another vector space S_2 (N_2 dimensional) on tensor product lead to a new N_1N_2 dimensional vector space and is denoted as $S_1 \otimes S_2$. Let $|X^{(1)}>$ and $|X^{(2)}>$ are kets of S_1 and S_2, respectively, and let $\hat{\mathbf{O}}^{(1)}$ and $\hat{\mathbf{O}}^{(2)}$ are operators of S_1 and S_2 , respectively. Further, let the following relations hold good in the S_1 and S_2 spaces.

$$\hat{\mathbf{O}}^{(1)}|X^{(1)}> \; = \; |Y^{(1)}> \quad . \tag{EC-7}$$

$$\hat{\mathbf{O}}^{(2)}|X^{(2)}> \; = \; |Y^{(2)}> \quad . \tag{EC-8}$$

For understanding the behavior of kets and operators associated with the tensor product, the following points are helpful.

(i) The product of kets $|X^{(1)}>$ and $|X^{(2)}>$ to be denoted as

$$|X^{(1)}X^{(2)}> \ \equiv \ |X^{(1)}> \ |X^{(2)}> \ , \qquad \text{(EC-9)}$$

belongs to the product space.

(ii) The product ket satisfies commutative property, i.e.,

$$|X^{(1)}X^{(2)}> \ = \ |X^{(2)}X^{(1)}> \ .$$

(iii) The product kets satisfy the distributive property with respect to sum. Thus, if

$$|X^{(1)}> \ = \lambda\, |X_a^{(1)}> \ + \mu\, |X_b^{(1)}> , \quad \text{then}$$

$$|X^{(1)}X^{(2)}> \ = \ \lambda\, |X_a^{(1)}X^{(2)}> \ + \mu\, |X_b^{(1)}X^{(2)}> \ .$$

where λ and μ are some constants.

(iv) Corresponding to each linear operator $\hat{\mathbf{O}}^{(1)}$ in space S_1 there exists a similar operator $\hat{\mathbf{O}}^{(1)}$ in the product space such that its action on the ket of S_1 happens as if $\hat{\mathbf{O}}^{(1)}$ belongs to S_1 space, and its action on the ket of S_2 space happens such that the ket of S_2 behaves as a constant. Thus in context with Eq.(EC-7) we can write

$$\hat{\mathbf{O}}^{(1)}|X^{(1)}\, X^{(2)}> \ = \ |Y^{(1)}\, X^{(2)}> \ .$$

Similarly, in context with Eq.(EC-8), we can get

$$\hat{\mathbf{O}}^{(2)}|X^{(1)}\, X^{(2)}> \ = \ |X^{(1)}\, Y^{(2)}> \ .$$

We can also see the validity of the commutative nature of the product of $\hat{\mathbf{O}}^{(1)}$ and $\hat{\mathbf{O}}^{(2)}$, i.e.,

$$\hat{\mathbf{O}}^{(1)}\, \hat{\mathbf{O}}^{(2)}|X^{(1)}\, X^{(2)}> \ = \ |Y^{(1)}\, Y^{(2)}>$$

$$= \ |Y^{(2)}\, Y^{(1)}> \ = \ \hat{\mathbf{O}}^{(2)}\, \hat{\mathbf{O}}^{(1)}|X^{(1)}\, X^{(2)}> . \qquad \text{(EC-10)}$$

Appendix-C: Additional equations for section 2

(i) Eigen equations

The following eigen equations associated with operators $\hat{\mathbf{O}}_{\mathbf{A}}$ and $\hat{\mathbf{O}}_{\mathbf{W}}$ described in section 2 are worth noting.

$$\hat{\mathbf{O}}_{\mathbf{A}} |BX_1\rangle = zero|BX_1\rangle = 0 \quad . \tag{EC-11}$$

$$\hat{\mathbf{O}}_{\mathbf{W}} |BX_1\rangle = |BX_1\rangle \quad . \tag{EC-12}$$

$$\hat{\mathbf{O}}_{\mathbf{A}} |AX_2\rangle = |AX_2\rangle \quad . \tag{EC-13}$$

$$\hat{\mathbf{O}}_{\mathbf{W}} |AX_2\rangle = zero\ |AX_2\rangle = 0 \quad . \tag{EC-14}$$

$$\hat{\mathbf{O}}_{\mathbf{A}} |BX_2\rangle = zero|BX_2\rangle = 0 \quad . \tag{EC-15}$$

$$\hat{\mathbf{O}}_{\mathbf{W}} |BX_2\rangle = zero|BX_2\rangle = 0 \quad . \tag{EC-16}$$

(ii) Normalization and orthogonality relations

$$\langle X_1|X_1\rangle = 1\ , \quad \langle X_2|X_2\rangle = 1\ . \quad \langle\phi|\phi\rangle = 1\ . \tag{EC-17}$$

$$\langle A|A\rangle = 1\ , \quad \langle B|B\rangle = 1. \tag{EC-18}$$

$$\langle\psi|\psi\rangle = 1\ , \quad \langle\psi_1|\psi_1\rangle = 1\ , \quad \langle\psi_2|\psi_2\rangle = 1. \tag{EC-19}$$

$$\langle X_1|X_2\rangle = \langle X_2|X_1\rangle = 0\ . \tag{EC-20}$$

$$\langle A|B\rangle = \langle B|A\rangle = 0\ . \tag{EC-21}$$

$$\langle AX_1|BX_1\rangle = \langle BX_1|AX_1\rangle = 0\ . \tag{EC-22}$$

$$\langle AX_2|BX_1\rangle = \langle BX_1|AX_2\rangle = 0\ . \tag{EC-23}$$

$$\langle AX_1|BX_2\rangle = \langle BX_2|AX_1\rangle = 0\ . \tag{EC-24}$$

$$\langle AX_2|BX_2\rangle = \langle BX_2|AX_2\rangle = 0\ . \tag{EC-25}$$

Appendix-D: Explanation of the buy-or-not-to-buy experiment

In the text, it has been discussed that in the experiment of Tversky and Shafir (1992), the subjects (undergraduate students at Stanford University) were asked to imagine that at the end of the fall quarter they have just taken a tough examination, and an attractive Christmas vacation package to Hawaii at very low price is being offered to them. One group of 67 subjects (say group-1) was asked to imagine that they have passed the examination, another group of 67 (say group-2) was asked to imagine that they have failed the examination, and the third group of 66 (say group-3) was asked to imagine that the outcome of their examination is not known to them. In this experiment, Tversky and Shafir (1992) observed that 54% subjects from group-1, 57% subjects from group-2, and 32% subjects from group-3 were ready to buy the vacation package.

We shall here show that these data can be explained by relations similar to the interference equations derived and discussed in section (2.5).

In view of simplicity, the mathematical details have been presented in section 2 such that they correspond to a specific example of the two-stage gambling experiment. But it is easy to generalize those conclusions for many other situations. As an example, for the experiment related with buying the vacation, the notations A, B, X_1 and X_2 can be assigned following meaning: A corresponds to the decision of buying the vacation, B corresponds to the decision of not buying the vacation, X_1 corresponds to passing the examination, and X_2 corresponds to failing the examination. Thus, here, instead of win operator we shall have pass operator, and instead of accept operator we shall have buy operator. With these differences, and assuming that except for the difference in the knowledge regarding the outcome of the examination, statistically the three samples are same, the experimental results of Tversky and Shafir (1992) can be summarized as follows:

$$p(A|X_1) = 0.54, \quad \text{(EC-26)}$$

$$p(A|X_2) = 0.57, \text{ and} \quad \text{(EC-27)}$$

$$p(A) = 0.32 \quad . \quad \text{(EC-28)}$$

In addition to not buying, if we assume the postponement of buying the vacation in the category of not buying the vacation, then from the above data we get

$$p(B|X_1) = 1- p(A|X_1) = 0.46, \quad \text{(EC-29)}$$

$$p(B|X_2) = 1- p(A|X_2) = 0.43, \quad \text{and} \quad \text{(EC-30)}$$

$$p(B) = 1- p(A) = 0.68 \quad . \quad \text{(EC-31)}$$

Following the work of Yukalov and Sornette (2009a), for the purpose of an estimate, we may also take $p(X_1) = p(X_2) = 0.5$. This gives $p(AX_1) = 0.270$, and $p(AX_2) = 0.285$. From these values and $p(A) = 0.32$, we need

$$q_{int}(A) = - 0.235. \quad \text{(EC-32)}$$

This value is consistent with Eq.(47) which becomes

$$- 0.555 \le q_{int}(A) \le 0.555 . \quad \text{(EC-33)}$$

If we consider $\cos(\theta_2 - \theta_1)$ of Eq. (46) as an adjustable parameter, then we can say that the experimental data can be explained by assigning

$$\cos(\theta_2 - \theta_1) = - 0.424. \quad \text{(EC-34)}$$

This value of $\cos(\theta_2 - \theta_1)$ with Eq.(73) leads to $q_{int}(A) = - 0.235$ [see Eq.(EC-32)].

Thus here also we see that using quantum theory we are able to explain the results of buy-or-not-to-buy experiment of Tversky and Shafir (1992) which could not be explained by any classical formulation such as sure-thing principle of Savage (1954).

Appendix-E: Mathematical details related with the merger and acquisition problem

In section 3.7.1 it has been discussed that the state of mind of the acquiring firm can be expressed as

$$|\Psi_A\rangle = a_1 \,|\, K_A\rangle + a_2 \,|\, O_A\rangle \quad , \quad \text{(EC-35)}$$

where, $|a_1|^2 + |a_2|^2 = 1$. (EC-36)

In addition to the description of these states $| K_A>$ and $| O_A>$ as presented there, the states $| K_A>$ and $| O_A>$ can also be considered to satisfy the following:

(a) States $| K_A>$ and $| O_A>$ are orthonormal, i.e.,

$$<K_A | K_A> = 1, \text{ and } <O_A| O_A> = 1. \quad \text{(EC-37)}$$

$$<K_A | O_A> = <O_A|K_A > = 0 \ . \quad \text{(EC-38)}$$

(b) States $| K_A>$ and $| O_A>$ are eigenstates of the price operator $\hat{\mathbf{P}}_{\mathbf{A}}$ having eigenvalues p_aK_A and zero, respectively. Thus

$$\hat{\mathbf{P}}_{\mathbf{A}} | K_A> = p_a K_A | K_A>, \quad \text{and} \quad \text{(EC-39)}$$

$$\hat{\mathbf{P}}_{\mathbf{A}} | O_A> = \text{zero} | O_A> = 0 \quad . \quad \text{(EC-40)}$$

Using Eq.(6a) for the expected value (P^A) of the perceived price of stock A in this state of mind of the negotiators of firm A, we can write:

$$P^A = \text{Market value of A in state } |\Psi_A> = <\Psi_A| \hat{\mathbf{P}}_{\mathbf{A}} |\Psi_A> \ . \quad \text{(EC-41)}$$

Using Eqs.(EC-35) and (EC-37)-(EC-40), above equation leads to

$$P^A = \text{Market value of A in state } |\Psi_A> = |a_1|^2 p_aK_A \ .$$

In view of Eq.(EC-36), above equation becomes,

$$P^A = (1-|a_2|^2)p_aK_A \ . \quad \text{(EC-42)}$$

Vitae

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